$$a_{mm} = -\tfrac{1}{2}(\mathbf{Y}_m, \delta \mathbf{M} \mathbf{Y}_m) \qquad (10)$$

Hence, the relation (7) together with the expressions (9) and (10) gives the variation of the eigenfunction, corresponding to eigenvalue λ_m, in terms of initially calculated quantities.

If the system is nonself adjoint, a_{mm} is determined by observing that we can normalize, to one, one of $\mathbf{Y}_m$'s at a specific point before and after variational procedure; if we do so and using Eq. (7) we get

$$\delta Y_m(x_o) = \sum_j a_{mj} Y_j(x_o) \qquad (11)$$

from which we obtain

$$a_{mm} = -\sum_{m \neq j} a_{mj} \mathbf{Y}_j(x_o)$$

IV. Applications

Example A)—Sturm-Liouville Problems

As an illustrating specific example consider the class of Sturm-Liouville systems resulting from the following functional [8]:

$$\lambda = \int_a^b [p(x)y'^2 - q(x)y^2]\, dx \Big/ \int_a^b r(x)y^2\, dx \qquad (12)$$

The aim is the calculation of changes in eigenvalues and eigenfunctions of this class of problems arising from the change of functions $p(x)$, $q(x)$, and $r(x)$ if we denote an arbitrary change of these functions by $\delta p(x)$, $\delta q(x)$, and $\delta r(x)$, respectively, and by designating the associated eigenvalue and eigenfunctions' changes by $\delta\lambda_m$ and δy_m, and utilize Eq. (5) we get

$$\delta\lambda_m = -\int_a^b (\delta p y_m'^2 + \delta q y_m^2 + \lambda_m \delta r y_m^2)\, dx \qquad (13)$$

and by using Eqs. (9) and (10) we obtain

$$a_{mj} = -\frac{1}{\lambda_m - \lambda_j} \int_a^b (\delta p y_m' y_j' + \delta q y_m y_j + \lambda_m \delta r y_m y_j)\, dx \qquad (14)$$

$$a_{mm} = -\frac{1}{2}\int_a^b \delta_r y_m^2\, dx$$

As a more specific example of a physical problem governed by the above class of equations we consider the longitudinal vibrations of a fixed-free rod. Starting with a uniform and homogeneous rod we would like to investigate the effect of the change in rod density on its natural frequencies and normal modes of vibrations. The natural frequencies and natural modes of the unperturbed (uniform) rod are

$$\lambda_m = \tfrac{1}{4}(m\pi)^2, \quad Y_m = \sin(m\pi/2)x, \quad m = 1, 3, 5, \ldots \qquad (15)$$

if the variation in density is denoted by the nondimensionalized function $\rho(x)$ the governing equations of time harmonic motion of density-perturbed rod is

$$d^2 Y/dx^2 + [1 + \rho(x)]\lambda Y = 0 \qquad (16)$$

where x is the length parameter, Y is the longitudinal displacement, and λ is the square of natural frequency; all in nondimensionalized quantities. According to Eq. (13) the variation in the natural frequency due to variation of the density is

$$\delta\lambda_m = -\int_o^1 \lambda_m \rho(x) Y_m^2\, dx \qquad (17)$$

Let, for the sake of illustration, $\rho(x) = \varepsilon x$, then Eq. (17) yields

$$\delta\lambda_m = -\varepsilon/2(\tfrac{1}{2} + 1/m\pi)\lambda_m = -\varepsilon/8(\tfrac{1}{2} + 1/m\pi)m^2\pi^2, \quad m = 1, 3, 5, \ldots$$

Similarly by using Eq. (14) we get

$$a_{mj} = 0, \quad m \neq j, \quad a_{mm} = -\varepsilon/4(\tfrac{1}{2} + 1/m\pi)$$

so

$$\delta Y_m = -\varepsilon/4(\tfrac{1}{2} + 1/m\pi) Y_m$$

Example B)—Sensitivity Analysis of Optimal Systems

The results of Secs. II and III may have application in the sensitivity study of the optimized systems governed by eigenproblems. A question of interest is to determine the variation in characteristic quantities due to the change of parameters and functions of a physical problem which are obtained in such a way as to optimize a certain response of the system. For illustration consider the problem of a column whose shape is determined such that among all the columns having the same length and volume has the highest buckling load. If x, λ, A, ϕ represent, respectively, the nondimensional axial coordinate, the buckling load, the variable area, and the bending moment then the governing boundary value problem for a clamped-crampel column is

$$\phi_{xx} + \lambda A^{-2}\phi = 0, \quad \begin{matrix} \phi_x(0) - \phi_x(1) = 0 \\ \phi_\xi(0) + \phi(0) - \phi(1) = 0 \end{matrix}$$

To achieve an optimum shape the following relation must hold[9] $\phi^2 = A^3$ and the resulting optimum shape comes out to be

$$A(x) = A_o \sin^2\theta(x)$$

with

$$\theta - \tfrac{1}{2}\sin 2\theta + \pi/2 = 2(\lambda/3)^{1/2} A_o^{-1} x$$

and

$$A_o = 1/\pi(\lambda/3)^{1/2}$$

the corresponding first buckling load and bending mode are

$$\lambda = 16\pi^2/3, \quad \phi(x) = A_o^{3/2}\sin^3\theta(x)$$

Now, let the optimum area be given a variation of the form

$$\delta A(x) = \varepsilon \sin \pi x$$

then, from Eq. (13) the resulting variation in the buckling load is found to be

$$\delta\lambda = -(4/\pi)\varepsilon\lambda$$

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Instability of Combustion

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THE following equations governing the phenomenon of intrinsic instability of combustion, leading to low frequency oscillations in a rocket motor using a single liquid propellant, were derived and investigated by L. Crocco[1–3]

Received September 24, 1973.

Index category: Combustion Stability, Ignition, and Detonation.

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$$M[d\phi(t)/dt]+\phi(t)+1 = 1-[d\tau(t)/dt] \quad (1)$$

and

$$\int_{t-\tau(t)}^{t} g[\phi(t')]\,dt' = A(\text{constant}) \quad (2)$$

In the above equations, ϕ is the nondimensionalized pressure in the rocket motor

$$\phi = [p(t)-\bar{p}]/\bar{p} \quad (3)$$

where bar denotes the mean value; M is the ratio of the mass of the hot air in the rocket motor to the mean mass flow rate; $g(\phi)$ is a positive monotonically increasing function of ϕ representing the rate of heat transfer from the combustion gas to the liquid propellant; and τ is the variable time lag.

Equations (1) and (2) can be combined to give

$$M\frac{d\phi(t)}{dt} + \phi(t) = \frac{g[\phi(t)]-g\{\phi[t-\tau(t)]\}}{g[\phi(t)-\tau(t)]} \quad (4)$$

It was shown that, if the equation is linearized, a necessary condition for unconditional stability, that is, stability for any value of τ, is given by

$$\left.\frac{dg(\phi)}{d\phi}\right|_{\phi=0} < \tfrac{1}{2}g(0) \quad (5)$$

This condition is sufficient only for infinitesimally small disturbances. In what follows, a sufficient condition for finite disturbances is derived.

It can be seen that the steady-state solution of Eq. (4) is $\phi = 0$. Let us consider a case where the system is disturbed for $t < 0$ such that

$$-1 < -\phi_m \leqq \phi(t) \leqq \phi_m, \quad \text{for} \quad t \leqq 0 \quad (6)$$

Regarding the behavior of $\phi(t)$ for large t, there are three possibilities; namely; 1) $\phi \geqq 0$, for all t greater than some T_1, 2) $\phi \leqq 0$, for all t greater than some T_2, or 3) ϕ is an oscillating function.

If the first condition is satisfied, then it is seen from Eq. (4) that

$$g[\phi(t-\tau)] \leqq \{g[\phi(t)]/[1+\phi(t)]\} \quad t > T_1 \quad (7)$$

If $g(\phi)$ is such that

$$(d/d\phi)[g(\phi)/(1+\phi)] \leqq 0, \quad \text{for} \quad |\phi| < \phi_m \quad (8)$$

then the right-hand side of the equality is a decreasing function of time whereas the left hand is an increasing function. They must, therefore, tend to their common limit, $g(0)$.

If the second condition is satisfied, then from Eq. (4)

$$g[\phi(t-\tau)] \geqq \{g[\phi(t)]/[1+\phi(t)]\}, \quad \text{if} \quad \phi > -1 \quad (9)$$

and

$$g[\phi(t-\tau)] \leqq \{g[\phi(t)]/[1+\phi(t)]\}, \quad \text{if} \quad \phi < -1 \quad (10)$$

Since $g > 0$, clearly, $\phi > -1$. This is expected because, $\phi < -1$ means negative pressure. If condition (8) is satisfied, the left-hand side of inequality (9) is a decreasing function of time, whereas the right-hand side is an increasing function. Hence, ϕ must tend to zero as before.

Suppose, ϕ is an oscillating function. Let the first local maxima or minima of $\phi(t)$ for $t > 0$, occur at $t = t_1$. Then for $0 \leqq t \leqq t_1$, we have $\phi_2(t) \leqq \phi(t) \leqq \phi_1(t)$, where ϕ_1 and ϕ_2 are defined as follows:

$$M(d\phi_1/dt)+\phi_1 = \{[\![g[\phi_1(t)]-g(-\phi_m)]\!]/g(-\phi_m)\} \quad (11)$$

and

$$M(d\phi_2/dt)+\phi_2 = \{[\![g[\phi_2(t)]-g(g_m)]\!]/g(\phi_m)\} \quad (12)$$

with $\phi(0) = \phi_1(0) = \phi_2(0)$. The steady-state solutions of Eqs. (11) and (12), ϕ_n and $+\phi_k$, respectively, are given by

$$[g(\phi_n)/(1+\phi_n)] = g(-\phi_m) \quad (13)$$

$$[g(\phi_k)/(1+\phi_k)] = g(\phi_m) \quad (14)$$

Let $f(\phi)$ be a continuous monotonically increasing function which satisfies the relations

$$[f(\phi)/(1+\phi)] = f(-\phi), \quad \text{for} \quad \phi \geqq 0 \quad (15)$$

$$f(0) = g(0) \quad (16)$$

Let g be bounded by f and $g(0)$, such that

$$|g(\phi)-g(0)| \leqq |f(k\phi)-g(0)| \quad (17)$$

where $k < 1$. Then we find

$$[g(\phi)/(1+k\phi)] < g(-\phi), \quad \text{for} \quad \phi > 0 \quad (18)$$

and

$$[g(\phi)/(1+k\phi)] > g(-\phi), \quad \text{for} \quad \phi < 0 \quad (19)$$

Under these conditions Eqs. (13) and (14) imply that

$$|\phi_k| < \phi_m k; \qquad |\phi_n| < \phi_m k \quad (20)$$

Hence

$$|\phi(t_1)| < k\phi_m \quad (21)$$

Similarly, at the next maxima or minima point, $t = t_2$, we have

$$|\phi(t_2)| < k^2\phi_m \quad (22)$$

Hence, ϕ must tend to zero.

In case $g(\phi)$ is linear over the range, $-\phi_m$ to ϕ_m, the sufficient condition reduces to

$$(dg/d\phi) < [g(0)k/(2+\phi_m k)] \quad (23)$$

For infinitesimal disturbances, the condition is given by

$$\left.\frac{dg}{d\phi}\right|_{0^+} + \left.\frac{dg}{d\phi}\right|_{0^-} < g(0)k \quad (24)$$

which reduces to condition (5) given by Crocco, if $dg/d\phi$ is continuous at $\phi = 0$.

Amplitude of Oscillations

Consider a system where $g(\phi)$ is a monotonically increasing function and that $[g/(1+\phi)]$ is a monotonically decreasing function. Let $dg/d\phi$ at $\phi = 0$ be sufficiently large so that the system is unstable.

Suppose there is a function $f(\phi)$, as defined by Eq. (15), such that

$$|g(\phi)-g(0)| \leqq |f(q\phi)-g(0)|, \quad \text{for} \quad \phi > -1 \quad (25)$$

where $q > 1$, and

$$|g(\phi)-g(0)| \leqq |f(k\phi)-g(0)|, \quad \text{for} \quad \phi_a \leqq \phi \leqq \phi_b \quad (26)$$

where $k < 1$. In order words, condition (17) is satisfied for a range $\phi_a < \phi < \phi_b$, but g is suitably bounded. In this case, it can be shown that if $\phi_b > q\phi_a$, the amplitude of self excited oscillations of the system is bounded by $|\phi| < q\phi_a$.

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An Improved Static Probe Design

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Introduction

PROBES for measuring in-stream static pressure in supersonic flows have been used for many years.[1] These probes have been shown to provide accurate and reliable measurement

Received October 23, 1973.
Index categories: Supersonic and Hypersonic Flow; Nozzle and Channel Flow.

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